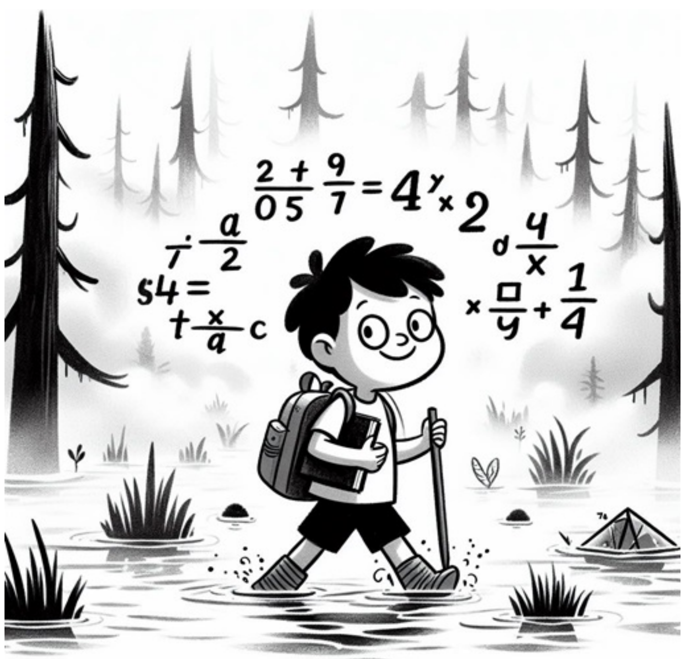
Math Problem Solving as a Quest



Dmitri Fomin

The Quest

There are two very common questions related to math problem solving. One of them is asked at the very beginning of the process:

"*How do I even start to solve this problem?*"

And the other one comes after the solution is revealed:

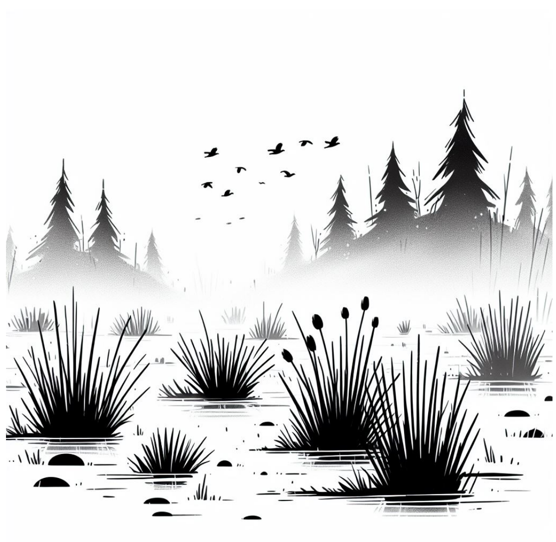
"*How was I supposed to come up with this solution?*"

* * *

Now allow me to describe something called "**The Problem Solving Quest**"... a more colloquial name for it would be "**Crossing the Swamp**".[1]

It is a useful way to describe and to visually present the process of math problem solving.

Imagine yourself standing in the middle of a large foggy swamp. You are on firm ground, on one of the little islets or mounds of dry land randomly scattered around the swamp.



There is an obvious goal, an objective that you are trying to achieve—you want to get out of the swamp, to reach the edge of the swamp, the dry land. If you are a fan of computer games, you can think of yourself as a computer game character playing some sort of a quest or a strategy game.

[1] The readers are highly encouraged to come up with other names which sound better in their native language.

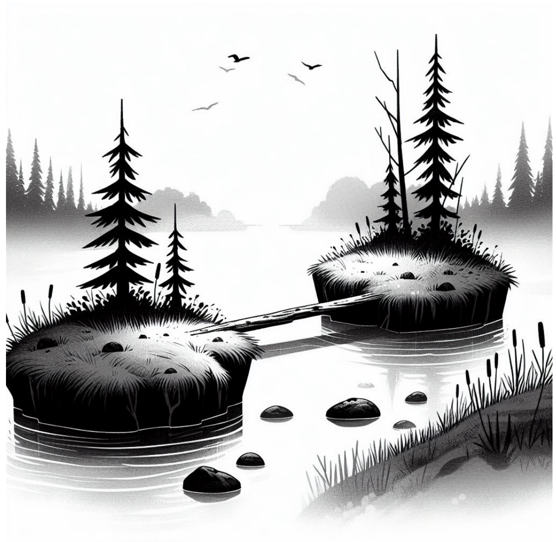
There are many useful analogies between this game and the process of problem solving. If you can see the nearest islets, then you can start jumping from one to another, trying to get closer to the target. Perhaps you can see that target, so you can immediately devise a plan; or perhaps it's too far away and you start moving in that general direction but without clear strategy.

Or you might find yourself in a situation when the next islet is too far and you simply can neither jump nor wade through the swamp to reach it. Then you need to make a decision: do I need this jump, or should I go back and try some other route?

And indeed, quite often you have to abandon your current approach and return back to one of the previous stages of your quest, where you can gather your thoughts, evaluate your experience and decide what to do next.

Only your experience will guide you to the right answer. And if you know that this jump is necessary but cannot make it on your own, then you need help. That help can be a tool (plank of wood, a long pole, stilts), or it can be some skill (learning how to jump higher and further, or how to see better through the fog).

Sometimes getting to the next islet is easy—perhaps there is a log connecting them, and you just need to walk on that log.



There are many, many analogies between the **Swamp Quest** and the process of solving a complicated problem. It can also be likened to climbing a mountain, exploring a cave system, or navigating a labyrinth—only your imagination is the limit.

Of course, not every problem can be immediately split into smaller steps. This also takes skill and experience, and sometimes, a stroke of luck.

In this article we use the set of questions surrounding one specific difficult problem, illustrating the Swamp Quest analogy as we go.

Problem Set # 1

In this text we will deal with finite collections (or **multisets**) of numbers. Hopefully, you already know what a collection is, but just in case, I want to emphasize the only difference between a collection and a set—a collection is allowed to contain several identical (equal) elements. When we write out a collection, we usually group the equal elements together.

This is a set: $A = \{1, -2.777, \pi, 2024\}$.

This is a collection: $A = \{1, 1, -2.777, \pi, \pi, \pi, 2024\}$.

It is important to remember that when we talk about sets or collections, there is no fixed order. So collection $\{1, 2, 4, 17, 17\}$ is the same as $\{17, 4, 1, 17, 2\}$.

$$\{1, 2, 4, 17, 17\} = \{17, 4, 1, 17, 2\}$$

Note. Collections are sometimes also called *bags* or *msets.*

* * *

We will begin first with a series of relatively simple problems. Their difficulty will gradually increase.

The first one is very easy.

A.1. Mathematician had three numbers. She computed all three of their pairwise sums and obtained a collection of three numbers $\{3, 4, 9\}$. What are the original three numbers?

Answer: A quick investigation shows that the original numbers are -1, 4 and 5. We will skip the proof leaving it to you as an easy exercise. ■

> The dry land is very close. You don't even have to jump, just a simple step through the muddy water and you are there!

Now, the obvious generalization of this question is:

A.2. From collection of three numbers a, b, and c we produced the collection of three possible pairwise sums: namely, $x = a + b$, $y = b + c$, and $z = a + c$.

$$\{a, b, c\} \longrightarrow \{x, y, z\} = \{a + b, b + c, c + a\}.$$

Can this operation be reversed? In other words, if we know the collection $\{x, y, z\}z$, then can we uniquely determine (restore) the original numbers a, b, c (as a collection, of course)?

Answer: Obviously, yes. Simply add together all the three given sums $a+b$, $b+c$, and $a+c$, obtaining the result which equals $2(a+b+c)$:

$$x + y + z = (a + b) + (b + c) + (c + a) = 2(a + b + c)\,.$$

Thus, by dividing $x+y+z$ by two, we can calculate the sum $\sigma = a+b+c$. Now, subtracting pairwise sums x, y, and z from σ, obviously, produces original numbers a, b, and c. ■

It is very important to emphasize that given the collection $\{x, y, z\}$ we do not know which of these three numbers represents which pairwise sum of the three original numbers. Please recall that there is no assigned order of the elements in a collection.

> This time the dry land was also quite close.
> One longer step and the quest is over!

Now, for the next step, we increase the size of the collection.

A.3. A collection of five numbers is given. The collection of ten possible pairwise sums of these numbers is

$$\{5, 7, 8, 9, 10, 12, 13, 14, 16, 18\}\,.$$

Find the original numbers.

Answer: $\{2, 3, 5, 7, 11\}$. **Hint**. Start with computing the sum of five original numbers—the result is 28. Since 5, obviously, is the sum of the two smallest original numbers, and 18 is the sum of the two largest ones, the middle number in the original collection must be equal to $28 - 5 - 18 = 5$. ■

> This time a real jump is required. But the swamp shoreline is always in sight, so it is easy to understand what has to be done.

A.4. Professor Smith wrote five numbers on five red cards. Then he took ten blue cards and wrote on them all ten possible pairwise sums of the original numbers.

$$\{a, b, c, d, e\} \longrightarrow \{a + b, a + c, \ldots, c + e, d + e\}\,.$$

He then shuffled the ten blue cards and gave them to Professor Jones. From this deck of cards can Jones determine the original five numbers?

Answer. Yes, he can. Denote the numbers by a, b, c, d, and e. For simplicity sake, and without loss of generality, assume that

$$a \leqslant b \leqslant c \leqslant d \leqslant e\,.$$

As before, Jones begins with computing the sum of the original collection, $\sigma = a + b + c + d + e$. This will allow Jones to find number c.

$$c = \sigma - (a + b) - (d + e)\,.$$

Since he knows that the second smallest number is equal to $a + c$, he can determine a. Similarly, he determines number e. The next step is to find number b from knowing the sum $a+b$, and similarly, finding the value of d from knowing the sum $d + e$. ■

The first jump is more or less the same as before. However, then we need a different approach—like using a long wooden plank.

A.5. A collection of four numbers is given. This collection has six possible pairwise sums, and they are

$$\{3, 4, 5, 7, 8, 9\}\,.$$

Find the original numbers.

Answer: Alas, this time we run into a problem. Quick computation will show that the original collection is either $\{1, 2, 3, 6\}$ or $\{0, 3, 4, 5\}$. So this operation of producing the collection of pairwise sums cannot be uniquely reversed. ■

Proving that something cannot be done is also a solution. The result is not what we expected, but that happens in real life as well.

A.6. Student wrote down four numbers. He then computed all six pairwise sums of these numbers, and they are

$$\{1, 3, 6, 7, 10, 13\}\,.$$

Find the original numbers.

Answer: This one is even worse. The student must have made a mistake in his computations, because such a four-number collection simply does not exist.

Indeed, if the original collection is

$$a \leqslant b \leqslant c \leqslant d\,,$$

then the collection of pairwise sums is

$$a+b \leqslant a+c \leqslant \ldots \leqslant b+d \leqslant c+d\,.$$

This means that the sum of the smallest and the largest pairwise sums equals

$$(a+b)+(c+d) = a+b+c+d\,.$$

The same must be true about the second smallest and the second largest pairwise sums

$$(a+c)+(b+d) = a+b+c+d\,.$$

But in the six-number collection we are given here the sum of the smallest and the largest elements equals $1+13=14$, while the sum of the second smallest and the second largest numbers is $3+10=13$. ■

Looking for examples and counterexamples is similar to exploring what lies around you. Almost like drawing a map of the swamp (or at least of some portion of it).

A.7. Is it ever possible to restore (recover) a collection of four numbers, given the collection of their six pairwise sums?

Hint. Consider a six-number collection such as

$$\{1,1,1,1,1,1\} \quad \text{or} \quad \{3,5,6,6,7,9\}\,.$$

This time, because you already have a map, navigating the swamp is easy.

Moser's Problem

In 1957 Canadian mathematician Leo Moser submitted a small and quite elementary question to the American Mathematical Monthly magazine. It consisted of two items more or less identical to our problems **A.2**, **A.4** and **A.7** above.

However, in the solution to the problem he formulated its obvious generalization and posed the following question

Moser's Problem. From any collection A of n numbers we can generate the collection of $n(n-1)/2$ pairwise sums of the collection's elements, which we will denote by $A^{(2)}$. Is that operation reversible? In other words, given the collection $A^{(2)}$, is it always possible to restore collection A? Or, in more formal language, is it true that

$$\forall A, B \quad A^{(2)} = B^{(2)} \;\Rightarrow\; A = B\,?$$

Let us call number n *singular* if the above statement is false. We already know that number 4 is singular (so is, obviously, $n = 2$). Also 3 and 5 are not singular.

This could give us a hint that the answer to Moser's Problem is determined by the parity of n: that is, if n is odd, then we can indeed recover the original collection from the collection of its pairwise sums; and if n is even, then it can be impossible.

Question. Is it true that number $n > 1$ is singular if and only if n is even?

The following problem (once you prove it) answers that question.

B.1. Prove that it is always possible to recover a collection of six numbers, given the collection of their fifteen pairwise sums.

Now, this problem is not as easy as the one for $n = 5$ but it is still possible to solve it using elementary approach.

This one requires several jumps. However, they are relatively easy because you have done them all before. You just need to figure out the jump sequence—find the first islet to jump to, then the next, etc.

However, for larger values of n it becomes more and more difficult to find the answer to Moser's Problem. Still, some results can be proved via elementary means.

B.2. If number n is singular, then $2n$ is singular as well.

> This time, the jumps are still not very complicated, but there are several of them. Also one or two of those jumps can be more difficult.

This proves that all powers of two are singular numbers. Finally, we have the following fact, which presents the complete solution to the Moser's Problem.

B.3. Number $n > 1$ is singular if and only if n is a power of two.

> This is tough. Simple jumps are not enough, and all of your previous inventions and techniques are not helping. As a matter of fact, we cannot even see the next islet. Where do we jump and how?

Alas, the easy and elementary methods now become insufficient. This question cannot be solved by means which lie within the margins of the high school curriculum.

Solving the Moser's Problem

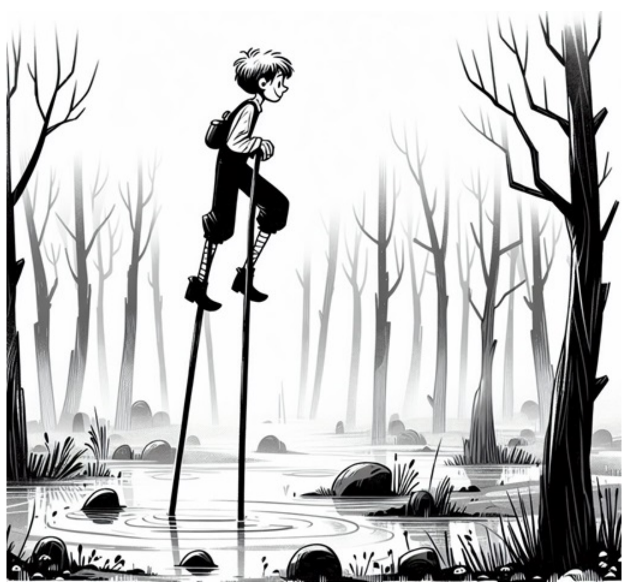
Let us stare at the problem for a little while. It is very likely that at some point your thoughts went like this:

How wonderful it would be if we knew which pair of numbers was actually added together to provide each specific sum!

For instance, what if we knew, for example, that the fifth number in the $A^{(2)}$ collection was precisely the sum of the second and the third numbers in collection A, and so on.

But... we do not know that. The numbers we have do not form an ordered sequence, they form a collection, where its elements can be shuffled around in any way, by any permutation. And here we get stuck... unless we find some way to deal with collections. Perhaps we read something or learned some areas of mathematics where a collection of numbers (without any order, and with possible repetitions) is often encountered.

Only your previous experience, as well as the toolkit (set of skills) that you have under your belt, can come to your help here.



First of all, let us learn from our previous experience—i.e., from solving the simpler versions of this problem. Almost in all of them we have computed the sum of all the numbers we have at our disposal; that is, we have added up all the pairwise sums of the original numbers.

What is so special about this operation? If you think about it, it should become obvious—the sum of several numbers is an operation that does not depend on the order (or arrangement) of these numbers, but only depends on the entire set (to be more, precise, on the collection of the numbers).

So, if, for instance, we have the collection of six pairwise sums $s_1, \ldots, s_6$ of some four unknown numbers, we can compute their sum

$$f(s_1, s_2, s_3, s_4, s_5, s_6) = s_1 + s_2 + \cdots + s_6\,.$$

We can also compute their product $s_1 s_2 \cdots s_6$, or the sum of their squares

$$g(s_1, s_2, s_3, s_4, s_5, s_6) = s_1^2 + s_2^2 + \cdots + s_6^2\,,$$

or, more generally, any expression which is *symmetric* with respect to these six numbers, meaning that you rearrange these numbers in any order it turns into itself. For those of you familiar with functions, we are talking here about functions of six variables which do not change their values when their arguments are rearranged. For example, we would have equalities like these:

$$f(s_1, s_2, s_3, s_4, s_5, s_6) = f(s_3, s_1, s_6, s_5, s_2, s_4) = f(s_5, s_2, s_4, s_1, s_6, s_3) = \ldots$$

Simplest symmetric functions are algebraic expressions, the so-called **symmetric polynomials**. Here are several examples of them for two, three, or four variables (denoted by x, y, z, and t):

$$\begin{aligned}
f(x, y) &= x + y \\
g(x, y) &= xy \\
h(x, y, z) &= xy + yz + zx \\
p(x, y, z, t) &= x^3 + y^3 + z^3 + t^3 \\
q(x, y, z, t) &= xy + xz + xt + yz + yt + zt
\end{aligned}$$

So instead of dealing with a collection of numbers, we could investigate some symmetric functions computed for this collection.

Second, we need some tools that will allow us to perform such an investigation.

So let us hope that at some point in your recent past you have learned about polynomials and their roots. Otherwise you really have no chance to solve this problem.

You would have to be a genius of Euclid's or Euler's magnitude to come up with the following ideas by yourself, without at least basic familiarity with polynomials and permutations.

So, from our experience with polynomials, we can recall that the collection of numbers $A = \{a_k\}$ can be represented by one object—namely, by monic polynomial[2] whose roots are the elements of A.

For instance, if we have a collection of, say, three numbers

$$A = \{a_1, a_2, a_3\} = \{-1, 2, 5\} ,$$

then we can construct polynomial

$$f_A(x) = (x - a_1)(x - a_2)(x - a_3) = \\ (x + 1)(x - 2)(x - 5) = x^3 - 6x^2 + 3x + 10 .$$

In exactly the same manner, for any multiset $A = \{a_1, \ldots, a_n\}$ of n numbers we can construct polynomial

$$f_A(x) = (x - a_1)(x - a_2) \cdots (x - a_n) .$$

> We found a useful tool (or a gadget, call it whatever you like) that allows us to transform our problem into a different one by replacing the collection A of numbers with the polynomial whose roots form this collection. Thus we have moved to another island in the swamp... in hope that it will be easier to reach the dry land from this new location.

The coefficients of that polynomials, when expressed as functions of collection A, are the so-called elementary *symmetric polynomials* of variables a_i:

$$\begin{aligned} \varepsilon_1(a_1, a_2, \ldots, a_n) &= a_1 + a_2 + \cdots + a_n , \\ \varepsilon_2(a_1, a_2, \ldots, a_n) &= a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n , \\ &\cdots \\ \varepsilon_n(a_1, a_2, \ldots, a_n) &= a_1a_2 \cdots a_n , \end{aligned}$$

where $\varepsilon_k(a_1, a_2, \ldots, a_n)$ is the sum of all possible products of some k different variables out of $\{a_1, \ldots, a_n\}$. In other words,

$$f_A(x) = x^n - \varepsilon_1 x^{n-1} + \varepsilon_2 x^{n-2} - \cdots + (-1)^n \varepsilon_n .$$

[2] That means that the leading coefficient of the polynomial equals 1.

This is the so-called **Vieta's Theorem** for polynomials.

Expression $\varepsilon_k(a_1, a_2, \ldots, a_n)$ is clearly, a polynomial of degree k in n variables $a_1, a_2, \ldots, a_n$.

Once again, such polynomials are called symmetric because they are expressed symmetrically through the variables. This means that their values do not depend on the order of variables $a_1, a_2, \ldots, a_n$. Therefore, each of these polynomials is, in effect, a function of collection $A = \{a_i\}$, not just a function of the ordered sequence of numeric arguments $(a_1, a_2, \ldots, a_n)$.

Lemma 1. If two collections of n numbers $A = \{a_i\}$ and $B = \{b_i\}$ satisfy the equalities

$$\begin{gathered}\varepsilon_1(a_1, a_2, \ldots, a_n) = \varepsilon_1(b_1, b_2, \ldots, b_n)\ ,\\ \varepsilon_2(a_1, a_2, \ldots, a_n) = \varepsilon_2(b_1, b_2, \ldots, b_n)\ ,\\ \ldots\\ \varepsilon_n(a_1, a_2, \ldots, a_n) = \varepsilon_n(b_1, b_2, \ldots, b_n)\ ,\end{gathered}$$

then collections A and B are identical.

Proof. The equalities above imply that polynomials f_A and f_B are the same. But that means that their collections of roots must be the same as well. ■

We need to prove that for certain values of n it is true that knowing collection $A^{(2)}$ would allow us to uniquely restore the original collection A.

As the Lemma 1 shows it would be enough to prove that for any index $1 \leqslant k \leqslant n$ value $\varepsilon_k(A)$ can be determined by collection $A^{(2)}$.

However, it turns out that working with polynomials ε_k is not very easy. There are other symmetric polynomials which are way more convenient for the task at hand. They are called power-sum symmetric polynomials σ_k:

$$\sigma_k(a_1, a_2, \ldots, a_n) = a_1^k + a_2^k + \cdots + a_n^k\,.$$

> We are learning new techniques and skills which should enable us to move around the swamp with ease. They are more technically demanding, but at the same time they allow us to get to the places we could not reach before.

Lemma 2. Elementary polynomials ε_k can be expressed as functions of power-sum polynomials σ_k and vice versa.

Example. Consider case $n = 3$, variables x, y, z, and symmetric polynomials

$$\varepsilon_1 = x + y + z\,, \qquad \varepsilon_2 = xy + yz + zx\,, \quad \varepsilon_3 = xyz\,.$$
$$\sigma_1 = x^1 + y^1 + z^1\,, \quad \sigma_2 = x^2 + y^2 + z^2\,, \quad \sigma_3 = x^3 + y^3 + z^3\,.$$

Then it is very easy to verify the identities

$$\varepsilon_1 = \sigma_1\,,\ \varepsilon_2 = \frac{1}{2}(\sigma_1^2 - \sigma_2)\,,\ \varepsilon_3 = \frac{1}{6}(\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3)\,.$$

Proof. We will leave the general case to you as an exercise. Or you can simply read about it in almost any introductory book dedicated to polynomials or higher algebra (on the internet you can also go to Wikipedia page on the so-called **Newton's Identities**). ■

Lemma 3. For $1 \leqslant k \leqslant n$ value $\sigma_k(A^{(2)})$ can be expressed by the following formula

$$\sigma_k(A^{(2)}) = (n - 2^{k-1})\sigma_k(A) + P(\sigma_1(A), \ldots, \sigma_{k-1}(A))\,, \qquad \langle * \rangle$$

where P is some expression (actually, a polynomial) in values $\sigma_1(A), \ldots, \sigma_{k-1}(A)$. For $k = 1$ polynomial P is a zero function.

For example, if $k = 1$, then, obviously,

$$\sigma_1(A^{(2)}) = \sum_{i<j}^{n}(a_i + a_j) = (n-1)\sum_{i=1}^{n} a_i = (n-1)\sigma_1(A)\,.$$

Similarly, for $k = 2$ we have the identity

$$\sigma_2(A^{(2)}) = \sum_{i<j}^{n}(a_i + a_j)^2 = (n-1)\sum_{i=1}^{n} a_i^2 + 2\sum_{i<j}^{n} a_i a_j$$
$$= (n-1)\sum_{i=1}^{n} a_i^2 + \left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2 = (n-2)\,\sigma_2(A) + \sigma_1^2(A)\,,$$

which is not very difficult to prove.

Proof. Now use the Newton's binomial formula to prove the lemma for all greater values of i. ■

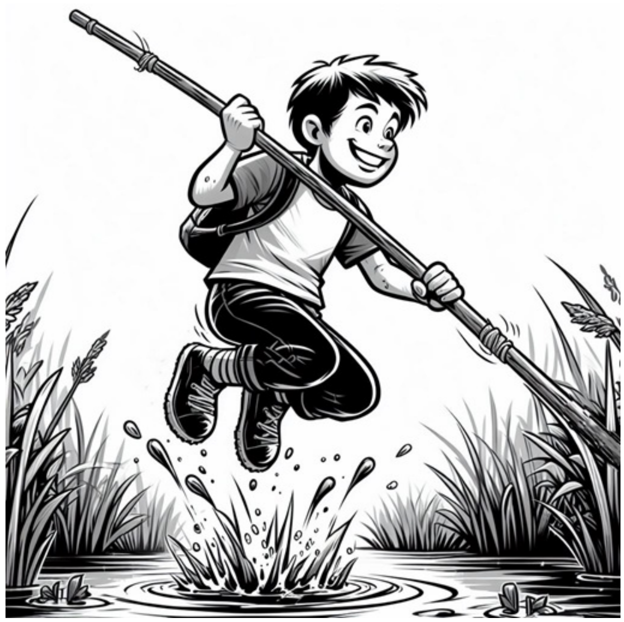
Hence, if n is not a power of 2, then coefficient $n - 2^{k-1}$ in $\langle * \rangle$ never vanishes (it never equals zero). Therefore, step by step, we can determine all the values $\sigma_k(A)$ based on the set of values $\sigma_k(A^{(2)})$. Therefore, it follows from Lemma 3 that for every number n which is not a power of 2, collection A can be uniquely restored from collection $A^{(2)}$.

After this you will have solved the original **Moser's Problem**.

The last two steps were not easy to prove. However, once we started using the "symmetric polynomials" tool, it was relatively easy to see that these steps were necessary for our solution.

Problem Set # 2

This set of questions is significantly more difficult then the first one.



Given collection A of n numbers we will call the sum of some s different elements of that collection[3] an s-*sum* of collection A.

Here are a few examples for collection $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$:

$$\text{3-sums}: \; a_2 + a_4 + a_7 \;,\; a_1 + a_5 + a_7 \;,\; a_4 + a_5 + a_6 \;,\; a_3 + a_4 + a_6 \;.$$
$$\text{4-sums}: \; a_1 + a_2 + a_5 + a_7 \;,\; a_2 + a_3 + a_4 + a_5 \;,\; a_3 + a_5 + a_6 + a_7 \;.$$

Generalized Moser's Problem. Let $1 \leqslant s \leqslant n$ be two natural numbers. For any given collection A of n numbers we can generate the collection of all s-sums of elements of A (there are $\binom{n}{s} = n!/s!(n-s)!$ of them), which we will denote by $A^{(s)}$. Is that operation reversible? In other words, given the collection $A^{(s)}$, is it always possible to determine collection A? Or, using more formal language, is it true that

$$\forall A, B \subset \mathbb{R} \quad |A| = |B| = n \;,\; A^{(s)} = B^{(s)} \;\Rightarrow\; A = B \;?$$

[3] They can be equal as numbers but they must be different as elements of A.

Let us call pair (n, s) *singular* if the above statement is false. For instance, it is pretty obvious that pair $(3, 3)$ is singular. Indeed, it is impossible to restore a 3-collection if you only know the sum of its three elements.

On the other hand, pair $(5, 4)$ is not singular, because adding up all 4-sums of a collection of five numbers allows us to find the sum σ of those five numbers. Then we simply subtract all 4-sums from σ to obtain the five original numbers.

We have already solved the Generalized Moser Problem for $s = 2$, and now we would like to try and find a solution for other specific values of parameter s.

Let us investigate $s = 3$. Then the version of Lemma 3 for $s = 3$ reads as follows.

Lemma 3′. For $1 \leqslant k \leqslant n$ value $\sigma_k(A^{(3)})$ can be expressed by the following formula

$$\sigma_k(A^{(3)}) = \frac{1}{2}\left(n^2 - n(2^k + 1) + 2 \cdot 3^{k-1}\right) \sigma_k(A) + P(\sigma_1(A), \ldots, \sigma_{k-1}(A)),$$

where P is some expression (actually, a polynomial) in values $\sigma_1(A), \ldots, \sigma_{k-1}(A)$. For $k = 1$ polynomial P is a zero function.

C.1. For $1 \leqslant k \leqslant n$ prove that polynomial

$$p_k(n) = n^2 - n(2^k + 1) + 2 \cdot 3^{k-1}$$

has integer roots if and only if $k \in \{1, 2, 3, 5, 9\}$. That is, the only polynomials p_i with integer roots are

$$\begin{aligned} p_1(n) &= (n-1)(n-2), \\ p_2(n) &= (n-2)(n-3), \\ p_3(n) &= (n-3)(n-6), \\ p_5(n) &= (n-6)(n-27), \\ p_9(n) &= (n-27)(n-486). \end{aligned}$$

This is very different from the original problem. Instead of dealing in some combinatorial arithmetic we are now exploring polynomial algebra and number theory. In other words, we find ourselves in a completely different swamp, which requires its own set of tools and skills.

We, of course, can discard the roots of polynomials p_k which are less than 3. Obviously, pair $(3,3)$ is singular. The only other possibly singular pairs of form $(n,3)$ are $(6,3)$, $(27,3)$, and $(486,3)$.

C.2. Find two different 6-collections A and B such that $A^{(3)} = B^{(3)}$.

> Begin with one single step (finding two different 3-collections with the same property). From that point you need to come up with one more jump.

This statement can be generalized as follows.

C.3. Prove that if $n = 2s$, then there exist two different n-collections A and B such that $A^{(s)} = B^{(s)}$.

> Use the same sequence of "moves" as in the previous quest.

The following is a very hard question to solve without help of a computer. However, it becomes relatively easy once we are allowed to use some computational package such as `Matlab`, `Mathematica`, or `SageMath`.

C.4. (a) Find two different 27-collections A and B such that $A^{(3)} = B^{(3)}$. (b) Find two different 486-collections A and B such that $A^{(3)} = B^{(3)}$.

> Unless you expand your toolkit (using computers) this is extremely difficult to do. Not impossible, but very, very difficult.

C.5. Formulate and prove version of Lemma 3 for $s = 4$, with the coefficient at $\sigma_k(A)$ being equal to the polynomial

$$p_k = \frac{1}{6}\left(n^3 - n^2(3 \cdot 2^{k-1} + 3) + n(2 \cdot 3^k + 3 \cdot 2^{k-1}) - 6 \cdot 4^{k-1}\right) .$$

> This quest is basically identical to one of those we did before. It only requires a bit more technical prowess.

C.6. For $1 \leqslant k \leqslant n$ prove that polynomial

$$n^3 - n^2(3 \cdot 2^{k-1} + 3) + n(2 \cdot 3^k + 3 \cdot 2^{k-1}) - 6 \cdot 4^{k-1}$$

has integer roots if and only if $k \in \{1, 2, 3, 4, 5, 6, 7\}$. That is, the only polynomials p_i with integer roots are

$$\begin{aligned}
p_1(n) &= (n-1)(n-2)(n-3)\,,\\
p_2(n) &= (n-2)(n-3)(n-4)\,,\\
p_3(n) &= (n-3)(n-4)(n-8)\,,\\
p_4(n) &= (n-4)(n^2-23n+96)\,,\\
p_5(n) &= (n-8)(n^2-43n+192)\,.\\
p_6(n) &= (n-12)(n^2-87n+512)\,.\\
p_7(n) &= (n-8)(n^2-187n+3072)\,.
\end{aligned}$$

> Here you need to make use of another helpful tool—namely, **modular arithmetic**.

C.7. Find two different 12-collections A and B such that $A^{(4)} = B^{(4)}$.

> Once again, very hard to do without computers.

C.8. If $n > 4$, $n \neq 8$ and $n \neq 12$, then collection A of n numbers can always be uniquely recovered from collection $A^{(4)}$.

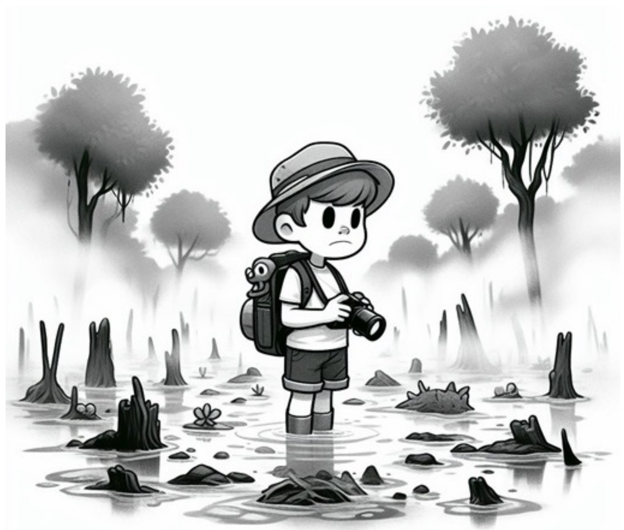
> This one is a direct corollary of the combination of several previous problems.

C.9. If $n > 5$, $n \neq 10$, then collection A of n numbers can always be uniquely recovered from collection $A^{(5)}$.

> This one is quite tough. It requires almost all of the tools and skills used in the solutions to the problems from this section.

Unsolved Problems

There are several very challenging unsolved questions and conjectures related to the Generalized Moser's Problem, including the general case of GMP itself.



D.1. The Generalized Moser's Problem. As of March 2024, GMP remains unsolved for $s \geqslant 6$.

This quest is likely to be as difficult as many other well-known problems in number theory and higher algebra (including the ones offered more than a hundred years ago).

For arbitrary value of s the following version of Lemma 3 was proved in 1962 by B. Gordon, A.S. Fraenkel, and E.G. Straus:

Theorem. For $1 \leqslant k \leqslant n$ value $\sigma_k(A^{(s)})$ can be expressed by the following formula

$$\sigma_k(A^{(s)}) = F_{s,k}(n)\,\sigma_k(A) + P(\sigma_1(A), \ldots, \sigma_{k-1}(A))\,,$$

where P is some expression (actually, a polynomial) in values $\sigma_1(A), \ldots, \sigma_{k-1}(A)$, and $F_{s,k}(n)$ is a polynomial of degree $s-1$ in variable n which can be expressed by the formula

$$F_{s,k}(n) = \sum_{m=1}^{s} (-1)^{m-1} m^{k-1} \binom{n}{s-m}\,.$$

Polynomials $F_{s,k}$ are called *Moser polynomials*, and the existence of their integer roots, as we now know, has direct relevance to the **GMP**.

* * *

The following conjecture, if proved, would allow us to resolve GMP for many small values of parameter s.

D.2. (**k-max Conjecture**) If $k \geqslant 2s$, then polynomial $F_{s,k}$ does not have integer roots.

> At least in this case the fog does not seem to be as thick as it is in the GMP quest.

D.3. (**Triplet Conjecture**) If $s > 2$ and $n > 2s$, then it is not possible to find three pairwise different n-collections A, B, and C, such that

$$A^{(s)} = B^{(s)} = C^{(s)}\,.$$

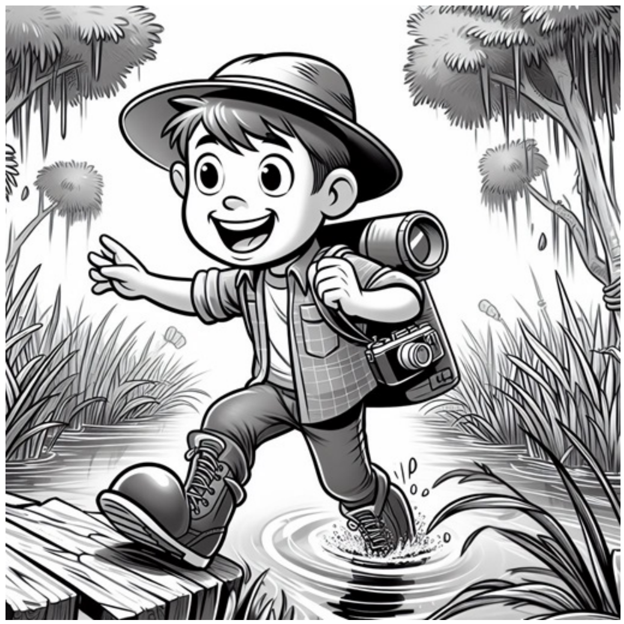
> I have no idea about the difficulty of this one.

* * *

For more details and more interesting problems, the readers are encouraged to review article **https://arxiv.org/abs/1709.06046**.

Conclusion

The Swamp Quest is NOT a problem solving method. It is merely a useful visual guide. Basically it provides you (or your teachers) with another interesting way to approach the entire challenging process of extracurricular mathematical education and math problem solving.



You can also use the Quest as a motivational tool. There are numerous analogies which you can use here—from bodybuilding to school sports to video games.

An obvious example: not knowing the next step of the solution is similar to the feeling of being lost when a thick fog envelopes your little islet.

Another one: knowing what needs to be done but being unable to do so is very similar to seeing the next islet but not having enough strength or skills to make the necessary jump.

The Quest helps to remind us that some things in life should not be discarded just because they are difficult. Many quests (or math problems) are easy and can be done within a few minutes. Others are not more difficult to figure out but simply take longer, because they involve several relatively easy steps. And other quests may take several hours or days to finalize because they require coming up with new, uncommon ways to overcome their obstacles.

And, of course, there are quests which can take years or even a lifetime—but we do them because they present challenges as riveting and fascinating as climbing Mount Everest, or conquering Venus and Mars. They teach us many lessons, one of which is that the worthwhile challenges should not be abandoned even if we sometimes cannot clearly see our way in the fog surrounding us.

* * *

So, when you get bored in the class solving countless nearly identical quadratic equations, tell yourself something like:

> *This is my gym practice![a] I am doing hundreds and thousands of squats and curl-ups to build up my muscles so that later it will be easy for me to jump from one swamp islet to another.*
>
> [a] If you are a video game aficionado, then this is your *grind*.

When your teacher recommends you a book or another resource to read, tell yourself:

> *This is my power-up! I am reading this book to acquire a very specific tool or a skill, so that later I will be able to see through the fog, or to soar above the swamp, or perhaps even to walk on water.*

And when someone asks you how you managed to solve a difficult problem while others had no clue, you can give them this answer:

> *I have spent hundreds (or even thousands) of hours in the gym (or practicing various levels of the game), learning how to use new tools and skills I acquired. No wonder that now I see things clearer, and I know more about how to attain my goals.*